AA214 – Fall 1996 – Problem Set 8

1. Subject: Transonic Potential Equations

   The Transonic Potential Equations or the Linear Cauchy-Riemann Equations are formed from the coupling of the steady compressible continuity equation of fluid dynamics

$$\partial_x \rho u + \partial_y \rho v = 0 \tag{1}$$

   and the vorticity definition

$$-\partial_x v + \partial_y u = 0 \tag{2}$$

   with vorticity $\omega = 0$ the irrotational potential assumption. Here ρ is density, u, v the Cartesian velocity components and the isentropic assumption leads us to

$$\rho = \left(1 - \frac{\gamma - 1}{2}\left(u^2 + v^2 - 1\right)\right)^{\frac{1}{\gamma-1}} \tag{3}$$

   with $\gamma = 1.4$ the ratio of specific heats. Pressure is defined as $p = \rho^\gamma$.

   The linear Cauchy-Riemann equations are recovered above and below by setting $\rho = 1$ instead of Eq 3.

   Combining Eq 1 and Eq 2 in vector form we have

$$\partial_x \mathbf{f}(\mathbf{q}) + \partial_y \mathbf{g}(\mathbf{q}) = 0 \tag{4}$$

   where

$$\mathbf{q} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} -\rho u \\ v \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} -\rho v \\ -u \end{pmatrix} \tag{5}$$

   Note: One approach to solving these equations is to cast them as a hyperbolic system where we solve

$$\partial_t \mathbf{q} + \partial_x \mathbf{f} + \partial_y \mathbf{g} = 0 \tag{6}$$

   (a) For Eq 6
      i. Find the flux Jacobians of $\mathbf{f}$ and $\mathbf{g}$.
      ii. Determine the eigenvalues and conditions (in terms of density and velocity) under which the system is hyperbolic. (Hint: A system is hyperbolic if the eigenvalues of it's flux Jacobians are real.)

   (b) The fluxes of the Euler equations are homogeneous of degree 1.
      i. Are the above fluxes $\mathbf{f}$ and $\mathbf{g}$ homogeneous of degree 1?, degree n?.
      ii. If we replace Eq 3 with $\rho = 1$, what can be said about the properties of the system?

2. Subject: Splitting/Factorization

   We shall use as the first example, the 1-Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 9 \tag{7}$$

   Let $u(0,t) = 0$ and $u(9,t) = 0$, so that we can simplify the boundary conditions. Assume below that second order central differencing is used, i.e.,

$$\frac{\partial^2}{\partial x^2} \approx \frac{1}{\Delta x^2}\mathcal{B}(n : 1, -2, 1) + O(\Delta x^3)$$

(a) *Space vector definition.* Assume a uniform grid $\Delta x = 1$ and $n = 10$ with the first point labeled 0 and the last point 9, that is, 8 interior points, 9 intervals.

i. What is the space vector for the natural ordering (monotonically increasing in index), $\overline{u}^{(n)}$? Only include the interior points. (Here the notation $\overline{u}^{(n)}$ refers to the natural ordering of the data, not the n^{th} iterative value).

ii. If we reorder the points with the odd points first and then the even points, write the space vector, $\overline{u}^{(*)}$?

iii. Write down the permutation matrices,(P_{n*},P_{*n}).

iv. The generic ODE representing the discrete form of Eq. 7 is

$$\frac{\partial \overline{u}^{(n)}}{\partial t} = [\mathcal{A}]^{(n)} \overline{u}^{(n)} + \overline{f} \tag{8}$$

Write down the matrix $[\mathcal{A}]^{(n)}$, then permute to the $\overline{u}^{(*)}$ form of Eq (2), defining the resulting ODE and $[\mathcal{A}]^{(*)}$. (Note $\overline{f} = 0$, due to the boundary conditions) (*Extra fun*: Show that $P_{*n}[\mathcal{A}]^{(n)} P_{n*} = [\mathcal{A}]^{(*)}$.)

v. Applying Implicit Euler time differencing, write the Delta form of the implicit algorithm. Comment on the form of the resulting implicit matrix operator.

(b) *System definition.* In Problem 2a, we defined $\overline{u}^{(n)}, \overline{u}^{(*)}, [\mathcal{A}]^{(n)}, [\mathcal{A}]^{(*)}, P_{n*}$, and P_{*n} which partition (sifts) the odd points from the even points. We can put such a partitioning to use. First define extraction operators

$$I^{(o)} = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} I_4 & 0_4 \\ \hline 0_4 & 0_4 \end{array}\right]$$

$$I^{(e)} = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} 0_4 & 0_4 \\ \hline 0_4 & I_4 \end{array}\right]$$

which extact the odd points or even points from $\overline{u}^{(*)}$.

We define $u^{(o)} = I^{(o)} \overline{u}^{(*)}$ and $u^{(e)} = I^{(e)} \overline{u}^{(*)}$ and start with our generic ODE, Eq. 8, in terms of $\overline{u}^{(*)}$ from Problem 2(a)ii.

i. Using the definitions of $u^{(o)}, u^{(e)}$, in the ODE (in terms of $\overline{u}^{(*)}$) and defining a splitting of $[\mathcal{A}]^{(*)} = [\mathcal{A}]^{(o)} + [\mathcal{A}]^{(e)}$. Write down the resulting ODE using the appropriate $\overline{u}^{(*)}$ notation.

It is useful at this point to rewrite

$$[\mathcal{A}]^{(*)} = \left[\begin{array}{c|c} D & U^T \\ \hline U & D \end{array}\right]$$

ii. Define D and U and $[\mathcal{A}]^{(o)}$ and $[\mathcal{A}]^{(e)}$ in terms of D and U.

iii. Apply Implicit Euler time differencing to the Split ODE of Problem 2(b)i. Write down the Delta form of the algorithm, the factored form and comment on the error terms.

iv. Examine the resulting implicit operators. Comment on their form. You should be able to argue that these are now trangular matrices (a lower and an upper). Comment on the solution process this gives us relative to the direct inversion of the resulting system from Problem 2(a)v. (Hint: You should get one term which looks like this:

$$\left[I - h[\mathcal{A}]^{(o)}\right] = \left[\begin{array}{c|c} I_4 - hD & 0_4 \\ \hline -hU & I_4 \end{array}\right]$$